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A Class of Topological Foliations on S^2 That Are Topologically Equivalent to Polynomial Vector fields

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Abstract. Let $\mathcal{F}$ be an oriented topological foliation on $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$ having only a finite number of singularities. If $\mathcal{F}$ has only a finite number of closed orbits and satisfies one additional condition, then it is shown that $\mathcal{F}$ is topologically equivalent to (the foliation induced by) a polynomial vector field.

1. **Introduction**

In this note we extend the main result of Schecter-Singer [4] from the C^1-class to the C^0-class. While the statement of our result is a little more general than that of [4], when we restrict to the C^1-class, the proofs given in [4] apply to the situation stated here (see Remark 2.1 below). Besides extending to the C^0-topology, we wanted to present, in a concise way, this very nice result of Schecter-Singer whose complete statement takes the first 16 pages of the referred article. We must say that this work depends on the results and arguments given in [4].

Two (one-dimensional) oriented topological foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, with or without singularities, defined on 2–manifolds M_1 and M_2, respectively, with corresponding set of singularities $S_1 \subset M_1$ and $S_2 \subset M_2$ are called topologically equivalent if there is a homeomorphism $h : M_1 \to M_2$ that takes S_1 onto S_2 and sends orbits (i.e. leaves) of $\mathcal{F}_1$ onto orbits of $\mathcal{F}_2$, preserving the direction of the orbits.

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In this paper, we consider a class of oriented topological foliation, with singularities, on S^2 that are topologically equivalent to (the foliations induced by) polynomial vector fields. Here $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. "Vector field on S^2" always means a tangent vector field to S^2; a polynomial vector field on S^2 is, in addition, one each of whose coordinates is a polynomial in x, y, z.

Isolated singularities p and q of oriented topological foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on 2–manifolds M and N ate called *topologically equivalent* if there are neighborhoods U and V of p and q such that $\mathcal{F}_1|_U$ is topologically equivalent to $\mathcal{F}_2|_V$ via a homeomorphism that takes p to q.

Orient S^2 by its unit outer normal vector field. *In this paper,* an isolated singularity p of an oriented topological foliation $\mathcal{F}$ on S^2 is said to be of finite type if (*i*) it is not topologically equivalent to a node; (*ii*) the local phase portrait of p is the union of finitely many hyperbolic, elliptic and parabolic sectors in the sense of [1, page 315]; in particular the elliptic sectors have no hyperbolic parts and the hyperbolic sectors have no elliptic parts ([3, Chapter VII – page 161]).

2. **Singularities of finite type**

Let p be an isolated singularity of an oriented topological foliation of finite type $\mathcal{F}$ on S^2. Then p has arbitrarily small *canonical neighborhoods* homeomorphic to compact discs whose boundaries are circles having the least possible number of tangencies with the foliation $\mathcal{F}$. In all figures, C will denote one of these circles [1, pp. 313-314], [3, Chapter VII – page 161]; see Fig. 1.

The restrictions of $\mathcal{F}$ to any two canonical neighborhoods of p are topologically equivalent. There is a familiar division of any canonical neighborhood of p into a finite number of elliptic, hyperbolic, and parabolic sectors [1, Chap. 8]; see Fig. 1. If γ is an orbit of $\mathcal{F}$, we shall denote by $\gamma(t)$ an arbitrary parametrization of γ, with t varying in $\mathbb{R}$ and such that, for increasing t, $\gamma(t)$ moves in conformity with the orientation of $\mathcal{F}$. The definitions below do no depend on the particular parametrization $\gamma(t)$ of γ. An α-(resp. ω-)*separatrix at* p is a semiorbit $\gamma(t)$ of $\mathcal{F}$ that approaches p as $t \to -\infty$ (resp. as $t \to \infty$) and that bounds a hyperbolic sector at p. We shall use the shorter expression *separatrix* to refer to an orbit of $\mathcal{F}$ that includes an α- or ω- separatrix at any singularity. If $\gamma = \gamma(t)$ is the orbit of $\mathcal{F}$ that passes through p at $t = 0$, then q belongs to the α-*limit set* (resp. ω-*limit set*) of p if and only if there is a sequence $t_n \to -\infty$ (resp. $t_n \to \infty$) such that $||\gamma(t_n) - q|| \to 0$. A *limit set* K is the α- or ω-limit set of some point; a limit set is always a compact connected union of orbits. Moreover, if $\mathcal{F}$ has only a finite number of singularities and closed orbits, then by the Poincaré Bendixson Theorem, each limit set of $\mathcal{F}$ is either a singularity or a single closed orbit or else a compact connected union of singularities and orbits that are α-separatrices at one end and ω-separatrices at the other. A limit set of the latter type is called a *separatrix cycle*. If Γ is an attracting separatrix cycle (resp. a repelling separatrix cycle), there exists an open cylinder A such that $A \cap \Gamma = \emptyset$, $\Gamma \subset \overline{A}$, and for all $p \in A$, the $\omega-$limit set of p is Γ (resp. the $\alpha-$limit set of p is Γ).

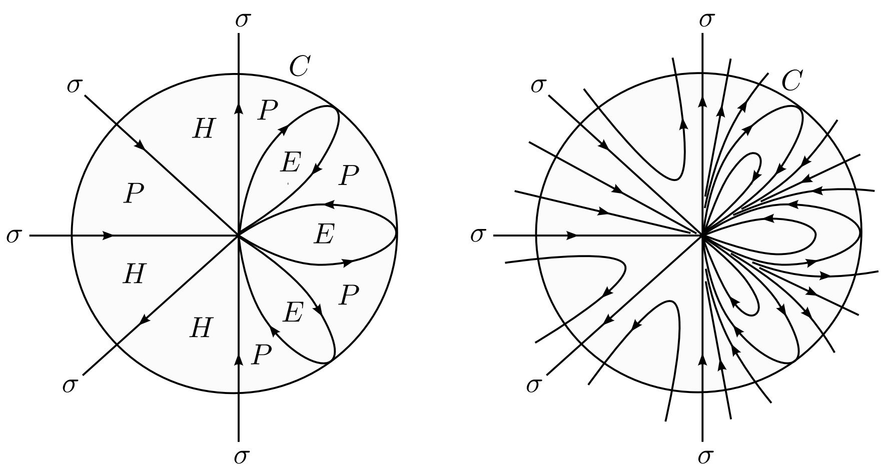


FIGURE 1. E=elliptic sector, H=hyperbolic sector, P=parabolic sector, σ=separatrix.

Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Given an subinterval I of $[0, \infty)$ we shall denote by

$$I \centerdot S^1 = \{(ax, ay) : a \in I,\ (x, y) \in S^1\}$$

The set $1 \centerdot S^1$ will be simply denoted by S^1. An oriented topological foliation $\mathcal{F}$ over $(0, 2) \centerdot S^1$ is said to be of type 1 (and degree $s \in \mathbb{N} \setminus \{0\}$) if (i) the terms of the sequence $\{p_k = (\cos(2\pi(k-1)/s), \sin(2\pi(k-1)/s)) : k = 1, 2, \ldots, s\}$ of S^1 make up the set S of singularities of $\mathcal{F}$; (ii) every such singularity p_k is topologically equivalent to either a hyperbolic saddle or to a node, and (iii) $S^1 \setminus S$ is made up of (full) orbits of $\mathcal{F}$. We shall say that $(p_1, p_2, \ldots, p_s)$ is the sequence of singularities of $\mathcal{F}$.

Let $\mathcal{F}$ be an oriented topological foliation on S^2. If p is an isolated singularity of finite type, there exists an open neighborhood V of p and a type one foliation $\mathcal{F}_1$ on $(0, 2) \centerdot S^1$ such that, for some $\varepsilon > 0$, $\mathcal{F}_1|_{(1,1+\varepsilon) \centerdot S^1}$ is topologically equivalent to $\mathcal{F}|_{V\setminus\{p\}}$. Since $\mathcal{F}_1|_{S^1}$ is a one–dimensional oriented foliation having only attracting and repelling singularities, it (and so $\mathcal{F}_1$) has an even number s of singularities. The foliation $\mathcal{F}_1$ will be said to be a topological blown up of p. Let $(p_1, p_2, \ldots, p_s)$ be the sequence of singularities of $\mathcal{F}_1$. The saddle-node sequence of $(\mathcal{F}_1, (p_1, p_2, \cdots, p_s))$, is the sequence of s symbols from the set $\{S_\alpha, S_\omega, N_\alpha, N_\omega\}$. The jth symbol is determined by the behavior of $\mathcal{F}_1$ in $[1, 2) \cdot S^1$ near p_j. The jth symbol of the saddle-node sequence is

- S_α (resp. S_ω) if there are two hyperbolic sectors of $\mathcal{F}_1$ at p_j in $[1, 2) \cdot S^1$, bounded by S^1 and an α- (resp. ω-) separatrix at p_j;
- N_α (resp. N_ω) if a neighborhood of p_j in $[1, 2) \cdot S^1$ is the union of negative (resp. positive) semiorbits of $\mathcal{F}_1$ that converge to p_j.

The saddle-node sequence of $(\mathcal{F}_1, (p_1, p_2, \cdots, p_s))$, will be said to be a saddle-node sequence of p. The *saddle-node cycle* of a singularity is just the saddle-node sequence thought of as a cycle: the first term in the sequence follows the last. In the following lemma, which is immediate, if δ denotes α (resp. denotes ω), then δ^* will denote ω (resp. will denote α).

LEMMA 2.1. *Let $\mathcal{F}$ be a topological foliation on S^2 having an isolated singularity p of finite type. Let $\mathcal{F}_1$ be a topological blown up of p and let $(p_1, p_2, \ldots, p_s)$ be the sequence of singularities of $\mathcal{F}_1$. Let $\Sigma = (\sigma_1, \sigma_2, \cdots, \sigma_s)$, be the saddle-node sequence of $(\mathcal{F}_1, (p_1, p_2, \cdots, p_s))$. Then, the first symbol in a saddle-node cycle $\Sigma = (\sigma_1, \sigma_2, \cdots, \sigma_s)$, of a finite type singularity p, can be taken to be S_α or N_ω. Moreover, for $\delta \in \{\alpha, \omega\}$,*

1. *S_δ (resp. N_δ) is always followed by S_{δ^*} or N_δ (resp. by S_δ or N_{δ^*}).*
2. *Each pair of consecutive terms σ_i, σ_{i+1} of the form S_δ, S_{δ^*} corresponds to exactly one hyperbolic sector $\mathrm{Sec}(\sigma_i, \sigma_{i+1})$ of p. See Fig. 2.*
3. *Each pair of consecutive terms σ_i, σ_{i+1} of the form N_δ, N_{δ^*} corresponds to exactly one elliptic sector $\mathrm{Sec}(\sigma_i, \sigma_{i+1})$ of p;*
4. *Let $\sigma_{i+1}, \cdots \sigma_{i+k}$ be a subsequence of Σ such that (i) $\sigma_{i+1}, \sigma_{i+k} \in \{S_{\delta^*}, N_{\delta^*}\}$ and, (ii) every term $\sigma_{i+2}, \cdots \sigma_{i+k-1}$ belongs to $\{S_\delta, N_\delta\}$ (and so S_δ's and N_δ's alternate). Then*

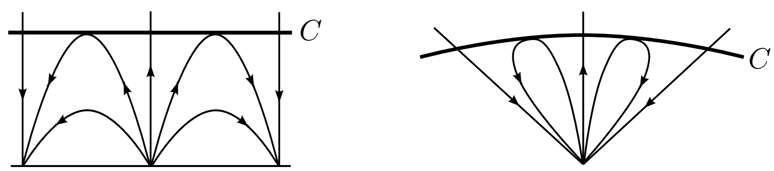
   (4.1) if $k \geq 3$ is odd and $\mathrm{Sec}(\sigma_{i+1}, \sigma_{i+2})$, $\mathrm{Sec}(\sigma_{i+k-1}, \sigma_{i+k})$ are elliptic then $\sigma_{i+1}, \cdots \sigma_{i+k}$ corresponds to exactly one parabolic sector $\mathrm{Sec}(\sigma_{i+1}, \cdots \sigma_{i+k})$ separating the refereed two elliptic sectors. See Fig. 3.

   (4.2) if $k \geq 4$ is even and one between $\mathrm{Sec}(\sigma_{i+1}, \sigma_{i+2})$, $\mathrm{Sec}(\sigma_{i+k-1}, \sigma_{i+k})$ is elliptic and the other hyperbolic, then $\sigma_{i+1}, \cdots \sigma_{i+k}$ corresponds to exactly one parabolic sector $\mathrm{Sec}(\sigma_{i+1}, \cdots \sigma_{i+k})$ separating the referred two sectors. See Fig. 4.

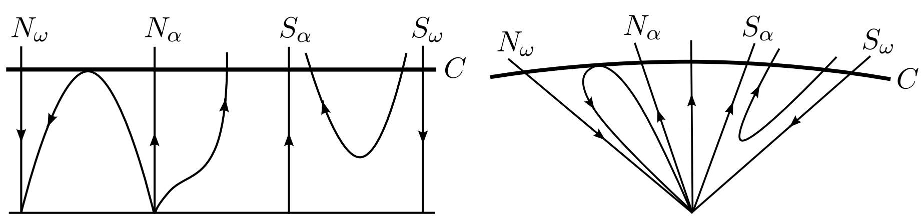
(4.3) if $k \geq 5$ is odd and $\mathrm{Sec}(\sigma_{i+1}, \sigma_{i+2})$, $\mathrm{Sec}(\sigma_{i+k-1}, \sigma_{i+k})$ *are hyperbolic then $\sigma_{i+1}, \cdots \sigma_{i+k}$ corresponds to exactly one parabolic sector* $\mathrm{Sec}(\sigma_{i+1}, \cdots \sigma_{i+k})$ *separating the referred two hyperbolic sectors. See Fig. 5.*

(5) *The topological blown up $\mathcal{F}_1$ of p can be taken so that, for any parabolic sector P of p, and modulo the restrictions imposed by (4) above, we may select the length k of the subsequence $\sigma_{i+1}, \cdots \sigma_{i+k}$ of Σ which satisfies* $P = \mathrm{Sec}(\sigma_{i+1}, \cdots \sigma_{i+k})$.

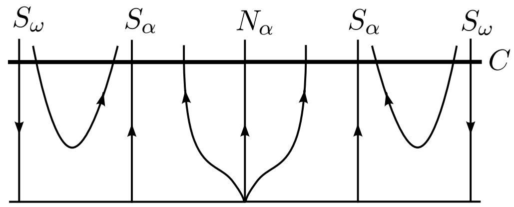
(a) FIGURE 2 (b)



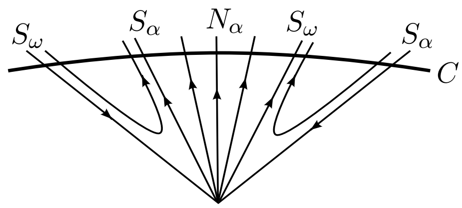
(a) FIGURE 3 (b)



(a) FIGURE 4 (b)



(a)



(b)

FIGURE 5

We shall say that the topological blown up $\mathcal{F}_1$ of the singularity p as above is *tight* if the subsequences of Σ associated to parabolic sectors have lengths 3,4 and 5 according as they correspond to the situations considered in (4.1), (4.2) and (4.3), respectively.

Let $\mathcal{F}$ be a oriented topological foliation on S^2 having finitely many singularities, each of which is either of finite type or topologically equivalent to a node. Let $\{p_1, p_2, \cdots, p_s\}$ be the singularities of $\mathcal{F}$ of finite type. For each such singularity p_i, we consider a topological blown up $\mathcal{F}_i$ of p_i and construct a corresponding saddle–node sequence $\Sigma_i = \sigma_{i1}\sigma_{i2}, \ldots, \sigma_{im_i}$ as above. Set $d_i = (m_i + 2)/2$. Each separatrix cycle K of $\mathcal{F}$ corresponds to a cycle C_K of some of the σ_{ij}. Any σ_{ij} in such a cycle is an S_α or an S_ω. Let $\mathcal{L}$ denote the set of all σ_{ij} such that $\sigma_{ij} \in \{S_\alpha, S_\omega\}$ and $\sigma_{ij+di-1} \in \{S_\alpha, S_\omega\}$. Here the second subscript is mod m_i. We say $(\mathcal{F}, (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_s))$ satisfies the *separatrix cycle condition* provided there is a function $f(\sigma_{ij})$ from $\mathcal{L}$ to the positive reals such that

(F1) $f(\sigma_{ij}) = f(\sigma_{ij+d_i-1})$ if $d_i - 1$ is even; $f(\sigma_{ij}) = [f(\sigma_{ij+d_i-1})]^{-1}$ if $d_i - 1$ is odd.

(F2) For every one-sided limit set K of $\mathcal{F}$ that is a separatrix cycle, either
  1. all σ_{ij} in C_K are in $\mathcal{L}$ and $\prod_{\sigma_{ij} \in C_K} f(\sigma_{ij}) > 1$ (resp. < 1) if K is attracting (resp. repelling); or
  2. some σ_{ij} in C_K are not in $\mathcal{L}$; if K is attracting (resp. repelling), all such σ_{ij} are S_α's (resp. S_ω's).

Our main result is

THEOREM 2.1. *Let $\mathcal{F}$ be a one-dimensional oriented topological foliation on S^2 such that*

(H1) it has only a finite number of closed orbits and it has finitely many singularities; every singularity is either of finite type or topologically equivalent to a node;

(H2) if $p_1, p_2, \cdots, p_s$, are the finite type singularities of $\mathcal{F}$, then, for every such p_i there exists a topological blown up $\mathcal{F}_i$ of p_i such that $(\mathcal{F}, (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_s))$ satisfies the separatrix cycle condition.

Then $\mathcal{F}$ is topologically equivalent to a polynomial vector field.

Remark 2.1. *S. Schecter and M. F. Singer state and prove the above theorem in the case that $\mathcal{F}$ is induced by a C^1-vector field and every $\mathcal{F}_i$ is a tight blown up of p_i. Nevertheless, within the C^1-class, their proof applies to the situation stated here. This fact was observed in* [4, Example 3 – page 423].

The proof of the following proposition follows immediately from the Smoothing Theorem and the Smoothing Corollary of [2].

Proposition 2.1. *Let $\mathcal{F}$ be a continuous one dimensional orientable foliation with singularities on the 2-sphere S^2. If the set of singularities of $\mathcal{F}$ is compact, then there exists a C^∞ vector field X on S^2 which is topologically equivalent to $\mathcal{F}$.*

Proof of Theorem 2.1. It follows from Proposition 2.1 that the exists a smooth vector field Y topologically equivalent to $\mathcal{F}$.

By Schecter-Singer main result [4] (see Remark 2.1) Y is topologically equivalent to a polynomial vector field □

References

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